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On Small Particles in Coagulation-Fragmentation Equations

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We consider a continuous coagulation-fragmentation equation, which describes the concentration $c(t, x)$ of particles of mass $x \in [0, \infty)$ at the instant $t \geqslant 0$ in a model where fragmentation and coalescence phenomena occur. We associate with this equation a nonlinear pure jump stochastic process, which is a stochastic microscopic description of the phenomenon. Existence is shown in a new case, where the total rate of fragmentation is infinite, and where we allow the presence of particles of mass 0. When coalescence is weaker than fragmentation, we study the appearance of particles of mass 0. We also show how to build solutions in the converse case where all particles at initial time have a mass 0. We finally study how the appearance of small particles leads to some regularization properties.

KEY WORDS: Coagulation-fragmentation equations; nonlinear stochastic differential equations.

1. INTRODUCTION

The coagulation-fragmentation equation deals with infinite particles system. In the model we consider, the particles are entirely determined by their mass. The main mechanisms taken into account are the coalescence of two clusters to form a larger one, and the breakage of clusters into two smaller ones. The notation $c(t, x)$ refers to the concentration of particles of mass $x \in [0, \infty)$ at the instant $t \geqslant 0$. This concentration c satisfies the continuous coagulation-fragmentation equation if:

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$$\left\{\begin{aligned} \frac{\partial}{\partial t} c(t,x) = & \frac{1}{2}\int_0^x K(y,x-y)\, c(t,y)\, c(t,x-y)\, dy \\ & - c(t,x)\int_0^\infty K(x,y)\, c(t,y)\, dy \\ & - \frac{1}{2} c(t,x) \int_0^x F(y,x-y)\, dy \\ & + \int_x^\infty F(x,y-x)\, c(t,y)\, dy, \\ c(0,x) = & \; c_0(x). \end{aligned}\right. \tag{1.1}$$

The coagulation kernel K is a symmetric function from $[0,\infty)^2$ into $\mathbb{R}_+$. $K(x,y)$ represents the rate of coalescence between particles of mass x and particles of mass y. The fragmentation kernel F is also a symmetric function from $[0,\infty[^2$ into $\mathbb{R}_+$, and $F(x,y)$ is the rate of fragmentation of particles of mass $x+y$ into particles of mass x, y.

Coagulation and fragmentation phenomena are plainly shown through equation (1.1) is quite natural. The first term of the right hand side member describes the appearance of particles of mass x by coalescence between particles of mass y and $x-y$. This phenomenon has a rate proportional to $K(y,x-y)$, but also to the concentrations $c(t,y)$ and $c(t,x-y)$. The factor $1/2$ avoids to count twice each pair $y, x-y$. The second term shows the disappearance of particles of mass x by coalescence of particles of mass x with any other particles.

The third term describes the disappearance of particles of mass x by breakage of particles of mass x into smaller particles of masses y and $x-y$: this phenomenon has a rate proportional to $F(y,x-y)$ and to the concentration $c(t,x)$. Finally, the last term explains the appearance of particles of mass x by fragmentation of larger particles.

We shall build a pure jump $\mathbb{R}^+$-valued Markov process $X=(X_t)_{t\geqslant 0}$ whose law is solution, in some sense, to (1.1). This process will represent the evolution of the mass of a typical particle. Thus, the process $(X_t)_{t\geqslant 0}$ is a stochastic microscopic description of the phenomenon described by the macroscopic equation (1.1). Our aim is to derive some properties of the solution to (1.1) from this stochastic process, by using probabilistic methods.

Laurençot[14] has results of existence of solutions to Eq. (1.1). He shows the existence of a possibly non conservative solution to (1.1), assuming mainly that F is continuous on $[0,\infty)^2$, that $K(x,y)\leqslant r(x)\,r(y)$ for some continuous function r on $[0,\infty)$, a condition meaning that the fragmentation is weaker than the coagulation, and the condition that

$\int_0^\infty (1+x)\, c_0(x)\, dx < \infty$. Laurençot's assumptions are not designed for small particles.

We should mention Norris,[15] who obtains an existence result in the pure coagulation case $F = 0$: he allows $K(x, y)$ to explode near 0, under an assumption meaning that the initial distribution is not too concentrated near 0. To be more precise, he makes the assumption that there exists a sub-linear function φ on $]0, \infty[$ such that $K(x, y) \leqslant \varphi(x)\, \varphi(y)$ and $\int_0^\infty \varphi^2(x)\, c_0(x)\, dx < \infty$.

Bertoin[2, 3] deals with pure fragmentation phenomena. The stochastic process he works on is a model for the evolution of all particles (while our process describes the evolution of one typical particle). He also allows multiple fragmentations and "erosion." Restricted to our context, Bertoin assumes that there is no coalescence $K = 0$, and that fragmentation is of the form $F(y, x-y)\, dy = x^\alpha \beta(y/x)\, dy$, where β is a function on $[0, 1]$, symmetric at $1/2$, and satisfying $\int_0^1 \theta(1-\theta)\, \beta(\theta)\, d\theta < \infty$. This is a particular case, but it allows the fragmentation kernel $F(y, x-y)$ to be non integrable: $\int_0^x F(y, x-y)\, dy = \infty$. Bertoin has noted the appearance of particles of mass 0. In the same framework, Haas[11] gives sufficient conditions for appearence and non-appearence of particles of mass 0. She also studies the time asymptotic behaviours of the concentration of particles of mass 0.

We finally mention Deaconu *et al.*[7] who introduced, in a pure coagulation case, a nonlinear stochastic process associated with Eq. (1.1).

Our paper is not relevent for large particles. It rather tries to extend Laurençot's existence results to the case where the fragmentation kernel $F(y, x-y)$ explodes at $y = 0$ for each x. Hence, our assumptions are designed to avoid gelation, i.e., the total mass will be preserved: for all $t \geqslant 0$,

$$\int_0^\infty xc(t, x)\, dx = \int_0^\infty xc_0(x)\, dx,$$

and we may assume without loss of generality that $\int_0^\infty xc_0(x)\, dx = 1$. Thus $Q_t(dx) = xc(t, x)\, dx$ is a probability measure on $[0, \infty)$ for each t. Rewriting Eq. (1.1) in terms of Q_t gives the following weak formulation: for any real function ϕ sufficiently regular on $[0, \infty)$, for every $t \geqslant 0$,

$$\begin{aligned} \langle Q_t, \phi \rangle = \langle Q_0, \phi \rangle &+ \int_0^t ds \int_0^\infty Q_s(dx) \int_0^\infty Q_s(dy) [\phi(x+y) - \phi(x)]\, \frac{K(x, y)}{y} \\ &+ \int_0^t ds \int_0^\infty Q_s(dx) \int_0^x dy [\phi(x-y) - \phi(x)]\, \frac{x-y}{x}\, F(y, x-y). \end{aligned} \tag{1.2}$$

We refer to ref. 7 (Definition 2.2) for more details on this (in the pure coagulation case). We should mention that Norris (when $F=0$)[15] and Haas (when $K=0$)[11] also consider a "measure" equation describing the evolution of $\mu_t(dx)=c(t,x)\,dx$.

Equation (1.2) allows singular initial conditions. Note also that if $F=0$, and if Q_0 has its support in $\mathbb{N}_*$, this equation is equivalent to the well-known discrete Smoluchowski coagulation equations. However, this equation does not contain the discrete coagulation-fragmentation equation, since the fragmentation is "continuous." Anyway, Jourdain[13] studied in detail the discrete case.

We would like to extend the probabilistic approach of ref. 7 in order to study the following points.

1. We want to extend Laurençot's existence results[14] to the case where the rate of fragmentation can be infinite; that is when $\lambda(x)=\int_0^x F(y,x-y)\,dy=\infty$. We thus consider the (smaller) quantity $\psi(x)=\frac{1}{x}\int_0^x y(x-y)\,F(y,x-y)\,dy$; $\psi(x)$ represents the rate of "loss of mass" of particles of mass x. An heuristic reasoning leads to the conclusion that even if $\lambda(x)=\infty$, a suitable control of ψ should suffice to produce non trivial solutions.

This case corresponds to phenomena where each particle has infinitely many breakages in any finite time interval, but very few loss of mass occurs at each breakage. This is linked with the ideas of Bertoin and Haas.[2, 11]

2. Such "infinite" fragmentation phenomena will lead, in some cases, to the appearance of particles of mass 0. We would like to obtain some conditions for this behaviour. Two natural questions are now: how does the coagulation takes into account the "mass" of particles of mass 0? In the case where coalescence is stronger than fragmentation, are there non trivial solutions to (1.2) when all the particles are initially of mass 0 (i.e., $Q_0=\delta_0$).

3. Finally, in the case where $\int_0^x F(y,x-y)\,dy=\infty$ for all $x>0$, does fragmentation give rise to a regularization property? In other words, if $\{Q_t(dx)\}$ is a solution to (1.2) with singular initial condition $Q_0(dx)$, is $Q_t(dx)$ absolutely continuous with respect to the Lebesgue measure dx (or eventually to $dx+\delta_0(dx)$) as soon as $t>0$?

Note that working with $Q_t(dx)=xc(t,x)\,dx$ (rather than $\mu_t(dx)=c(t,x)\,dx$) seems to be a good point of view for studying phenomena concerning the presence of particles of mass 0. Indeed, we shall prove that there is no loss of mass, but rather an infinity of particles of mass 0 which represents a positive total mass. Rigorously, this simply means that $Q_t(\{0\})>0$. From the other point of view, this should be written $0\times\mu_t(\{0\})>0$.

The rest of the paper is divided into three sections. In Section 2, we introduce our assumptions, and write down the generalised coagulation-fragmentation equation. We also introduce an associated pure jump non-linear stochastic process, continuing the work of ref. 7, inspired by Tanaka,[16] Graham and Méléard.[10] Section 3 is devoted to the exposition of our main results. The proofs are finally collected in Section 4.

2. A GENERALISED EQUATION

First of all, we state our assumptions.

Assumptions (A).

1. The coagulation kernel K is a continuous symmetric map from $[0, \infty)^2$ into $\mathbb{R}_+$. There exists a constant C such that

$$K(x, y) \leqslant C(1+x+y).$$

2. The fragmentation kernel F is a continuous symmetric map from $(0, \infty)^2$ into $\mathbb{R}_+$. It is also continuous from $[0, \infty)^2$ into $\mathbb{R}_+ \cup \{+\infty\}$. The following function is continuous on $[0, \infty)$

$$\psi(0) = 0; \qquad \psi(x) = \frac{1}{x} \int_0^x y(x-y)\, F(y, x-y)\, dy \qquad (x > 0).$$

There exist constants $p \geqslant 1$ and C such that

$$\psi(x) \leqslant C(1+x^p). \tag{2.1}$$

Finally, for all $\varepsilon \in (0, 1)$,

$$\lim_{n \to \infty} \sup_{x \in [\varepsilon, 1/\varepsilon]} \psi_n(x) = 0, \tag{2.2}$$

where

$$\psi_n(x) = \frac{1}{x} \int_0^x y(x-y)\, F(y, x-y)\, \mathbb{1}_{\{F(y, x-y) \geqslant n\}}\, dy.$$

3. The initial distribution $Q_0(dx)$ is a probability measure on $[0, \infty)$ ($Q_0(\{0\})$ can be positive), and $\int_0^\infty x^{p+1} Q_0(dx) < \infty$ where p is defined in (2.1).

Note that $\psi(x)$ represents the "rate of loss of mass" of particles of mass x. Hence, the assumption $\psi(0)=0$ is meaningful. Remark also that (2.2) is a technical condition, which does not seem very stringent.

With such assumptions, we have to consider a more general equation than (1.2), which takes into account the particles of mass 0.

We denote by $C_b^1([0,\infty))$ the set of C^1 functions with a bounded derivative. For a measure q and a function ϕ, the notation $\langle q, \phi\rangle$ stands for $\int \phi(x)\, q(dx)$.

Definition 2.1. Assume (A). A family $\{Q_t\}_{t\geqslant 0}$ of probability measures on $[0,\infty)$ solves (CF) if the following conditions hold:

(i) for all $T\geqslant 0$, $\sup_{[0,T]}\int_0^\infty x^p Q_t(dx)<\infty$ where p is defined in (2.1);

(ii) for all $\phi\in C_b^1([0,\infty))$, all $t\geqslant 0$,

$$\langle Q_t, \phi\rangle = \langle Q_0, \phi\rangle + \int_0^t ds[\langle Q_s, \mathscr{K}_{Q_s}\phi\rangle + \langle Q_s, \mathscr{F}\phi\rangle] \tag{CF}$$

where for any probability measure q on $[0,\infty)$, any x in $[0,\infty)$,

$$\begin{aligned}\mathscr{K}_q\phi(x) &= \int_0^\infty \frac{\phi(x+y)-\phi(x)}{y}\,K(x,y)\,\mathbb{1}_{\{y>0\}}q(dy)+\phi'(x)\,K(x,0)\,q(\{0\}),\\ \mathscr{F}\phi(x) &= \int_0^x [\phi(x-y)-\phi(x)]\,\frac{x-y}{x}\,F(y,x-y)\,dy.\end{aligned} \tag{2.4}$$

Note first that, thanks to our assumptions, $\lim_{x\to 0^+}\mathscr{F}\phi(x)$ is always 0, and that the integrability condition (i) suffices to ensure that every expression makes sense and is finite in (CF) and (2.4). Note also that if for each $s\geqslant 0$, $Q_s(dx)$ has a density $f(s,x)$, then $c(s,x)=f(s,x)/x$ is a solution, in a weak sense, to (1.1).

Following now the ideas of Tanaka,[16] Graham and Méléard,[10] about the Boltzmann equation, see also ref. 7 for the Smoluchowski coagulation equation, we associate with equation (CF) a nonlinear pure jump stochastic process. Some notations are needed to introduce such a process.

Notation 2.2. We consider two probability spaces: $(\Omega, \mathscr{G}, \{\mathscr{G}_t\}_{t\geqslant 0}, P)$ is an abstract space and $([0,1], \mathscr{B}[0,1], d\alpha)$ is an auxiliary space (here, $d\alpha$ denotes the Lebesgue measure). In order to avoid confusions, the expectation on $[0,1]$ will be denoted E_α, the laws $\mathscr{L}_\alpha$, the processes will be called α-processes.

We denote by $\mathbb{D}([0,\infty),\mathbb{R}_+)$ the space of càdlàg functions from $[0,\infty)$ into $\mathbb{R}_+$, and we endow this space with the Skorokhod topology, see Jacod and Shiryaev.[12]

Definition 2.3. Assume (A). $(X, \tilde{X})$ is a solution to the problem (SDE) if the following conditions are fulfilled:

(i) $X = \{X_t(\omega)\}_{t\geqslant 0}$ is an adapted process whose paths belong to $\mathbb{D}([0,\infty), \mathbb{R}_+)$, while $\tilde{X} = \{\tilde{X}_t(\alpha)\}_{t\geqslant 0}$ is an α-process such that $\mathscr{L}(X) = \mathscr{L}_\alpha(\tilde{X})$;

(ii) for all T, $E[\sup_{[0,T]} |X_t|^{p+1}] < \infty$;

(iii) $\mathscr{L}(X_0) = Q_0$;

(iv) there exist two independent Poisson measures $N(ds, d\alpha, du)$ and $M(ds, dy, du)$, adapted to $\{\mathscr{G}_t\}_{t\geqslant 0}$, on $[0,\infty)\times[0,1]\times[0,\infty)$ and $[0,\infty)\times[0,\infty)\times[0,\infty)$ with intensity measures $ds\, d\alpha\, du$ and $ds\, dy\, du$, such that the following nonlinear stochastic equation holds:

$$\begin{aligned} X_t = X_0 &+ \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha)\, \mathbb{1}_{\{\tilde{X}_{s-}(\alpha)>0\}} \mathbb{1}_{\left\{u\leqslant \frac{K(X_{s-},\tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds, d\alpha, du) \\ &+ \int_0^t K(0, X_{s-})\, P(X_{s-} = 0)\, ds \\ &- \int_0^t \int_0^\infty \int_0^\infty y\, \mathbb{1}_{\{y\in(0,X_{s-})\}} \mathbb{1}_{\left\{u\leqslant \frac{X_{s-}-y}{X_{s-}} F(y, X_{s-}-y)\right\}} M(ds, dy, du). \qquad \text{(SDE)} \end{aligned}$$

Note that X can be seen as the evolution of the mass of a typical particle. Hence equation (SDE) is quite comprehensible: the Poisson integral with N shows that sometimes, the mass of another particle $(\tilde{X}_s(\alpha))$ is added to the typical particle. This is done at the rate $K(X_{s-}, \tilde{X}_{s-}(\alpha))/\tilde{X}_{s-}(\alpha)$. The Lebesgue integral explains that particles of mass 0 (which total mass equals $P(X_{s-} = 0)$) coagulate "continuously" on the typical particle, with the rate $K(0, X_{s-})$. Finally, the Poisson integral involving M shows that at some instants, the typical particle breaks into two smaller particles; one of these two particles is chosen proportionally to its mass: we thus subtract y to X, for some $y \in (0, X_{s-})$, at the rate $F(y, X_{s-} - y)\, \frac{X_{s-}-y}{X_{s-}}$. We refer to ref. 7 for more details on a closely related topic. We now make explicit the link between (CF) and the nonlinear stochastic equation (SDE).

Remark 2.4. We assume (A) and consider a solution $(X, \tilde{X})$ to the problem (SDE). For each $t \geqslant 0$, we set $Q_t = \mathscr{L}(X_t) = \mathscr{L}_\alpha(\tilde{X}_t)$. Then the family $\{Q_t\}_{t\geqslant 0}$ is a solution to (CF) in the sense of Definition 2.1.

The proof of this remark is straightforward: it suffices to compute $E(\phi(X_t))$ for $\phi \in C_b^1$, by using the Itô formula for jump processes (see e.g., Jacod and Shiryaev,[12] p. 57). See ref. 7, Proposition 2.9 for a similar argument.

3. MAIN RESULTS

We begin by expose existence results for the problem (SDE) and hence for (CF). Then we show, when the fragmentation is null or finite, that equation (CF) is equivalent to standard discrete or continuous coagulation-fragmentation equations. Then we study the case where X_0 is a.s. 0, and we show that the solution X_t becomes positive provided the kernel K is positive. We give an explicit example of such a phenomenon. Then we state results about the hitting time of 0 under an hypothesis ensuring that fragmentation is involved sufficiently often. We finally present a result of regularization due to an infinite fragmentation rate.

3.1. Existence

We first give an existence result in the standard case where F is bounded.

Proposition 3.1. Assume (A), that F is a continuous bounded function on $[0, \infty)^2$, and that $\int_0^\infty x^{-1} Q_0(dx) < \infty$.

1. Then there exists a solution $(X, \tilde{X})$ to the problem (SDE), and this solution satisfies that for all $t \geqslant 0$, $P(X_t = 0) = 0$. Hence the drift term can be dropped in (SDE).

2. Hence there exists a solution $\{Q_t\}_{t \geqslant 0}$ to (CF) for which $Q_t(\{0\}) = 0$ for all $t \geqslant 0$. This solution is "standard," in the sense that it satisfies Eq. (1.2), for every $\phi \in C_b^1$.

Then we state a stability result.

Theorem 3.2. Assume (A). For each positive integer n, set $F_n(x, y) = F(x, y) \wedge n$ and $Q_0^n = Q_0([0, 1/n])\, \delta_{1/n}(dx) + \mathbb{1}_{[1/n, \infty[}(x)\, Q_0(dx)$. Then we know from Proposition 3.1 that for each n, there exists a solution $(X^n, \tilde{X}^n)$ to $(\text{SDE})_n$, where Q_0 and F have been replaced by Q_0^n and F_n in (SDE).

1. The sequence of laws $Q^n = \mathscr{L}(X^n) = \mathscr{L}_\alpha(\tilde{X}^n)$ is tight in the space $\mathscr{P}(\mathbb{D}([0, \infty), \mathbb{R}_+))$.

2. Any limiting point Q is the law of a solution $(X, \tilde{X})$ to (SDE) (i.e., $\mathscr{L}(X) = \mathscr{L}_\alpha(\tilde{X}) = Q$).

Hence there is existence of solutions for (SDE) and consequently for (CF).

This result shows the existence of solutions for (SDE) and (CF), but also that these solutions can be obtained by going to the limit in standard equations.

We finally present a particular case of fragmentation kernels which allow to rewrite (SDE) in a simpler form. Bertoin[2] considers this kind of fragmentation kernels.

Remark 3.3. Assume (A), and consider a solution $(X, \tilde{X})$ to (SDE). Assume that $F(y, x-y) = \alpha(x)\, \beta(y/x)$ for some nonnegative function α on $[0, \infty[$, and some nonnegative continuous function β on $(0, 1)$, symmetric at $1/2$. Note that if so, the function ψ satisfies

$$\psi(x) = x^2\alpha(x) \int_0^1 \theta(1-\theta)\, \beta(\theta)\, d\theta.$$

Then there exists an adapted Poisson measure $\mu(ds, d\theta, du)$ on $[0, \infty) \times [0, 1] \times [0, \infty)$ with intensity $ds(1-\theta)\, \beta(\theta)\, d\theta\, du$ such that $(X, \tilde{X})$ satisfies

$$X_t = X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha)\, \mathbb{1}_{\{\tilde{X}_{s-}(\alpha)>0\}} \mathbb{1}_{\left\{u \leqslant \frac{K(X_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds, d\alpha, du)$$

$$+ \int_0^t K(0, X_{s-})\, P(X_{s-} = 0)\, ds$$

$$- \int_0^t \int_0^1 \int_0^\infty \theta X_{s-}\, \mathbb{1}_{\{u \leqslant X_{s-}\alpha(X_{s-})\}}\, \mu(ds, d\theta, du). \tag{3.1}$$

3.2. The Finite Case

The first case we deal with is almost obvious.

Proposition 3.4. Assume (A), that $F = 0$, and that $Q_0(\mathbb{N}_*) = 1$. Consider a solution $(X, \tilde{X})$ to (SDE). Then for all $t \geqslant 0$, X_t belongs a.s. to $\mathbb{N}_*$. Hence, $n(k, t) = P(X_t = k)/k = Q_t(\{k\})/k$ is a solution to the standard discrete Smoluchowski coagulation equations, see, e.g., ref. 7.

The second result is also intuitively clear. It shows that at the intersection of our case with Laurençot's one,[14] we obtain the same equation.

Proposition 3.5. Assume (A), that $\int_0^\infty x^{-1} Q_0(dx) < \infty$, that $Q_0(dx) \ll dx$, and that the fragmentation is finite, in the sense that

$$\int_0^x F(y, x-y)\, dy \leqslant C(1 + x^{p+1}). \tag{3.2}$$

Consider a solution $(X, \tilde{X})$ to (SDE). Then for all $t \geqslant 0$, $Q_t = \mathscr{L}(X_t) \ll dx$. Thus, if we denote by $f_t(x)$ its density, then $c(t, x) = f_t(x)/x$ is a solution to equation (1.1), in a weak sense.

The two previous propositions would have been perhaps useful in the works of Norris,[15] see also ref. 7. Indeed, both deal with "measure" solutions. They prove in their papers that the way they build the solutions make them be discrete or absolutely continuous. We show here that any solution to (SDE) with X_0 discrete (resp. absolutely continuous) is discrete (resp. absolutely continuous).

3.3. Starting from Dust

We only treat the case where the fragmentation rate is finite for simplicity, but such results may also hold in the infinite case with a suitable control of ψ.

Proposition 3.6. Assume (A), that the whole mass is initially concentrated at 0: $Q_0 = \delta_0$. Assume also that $K(0, 0) > 0$, and that $\int_0^x F(y, x-y)\, dy \leqslant C(1+x^{p+1})$. Consider a solution $(X, \tilde{X})$ to (SDE).

1. Then for all $t > 0$, $P(X_t = 0) = 0$.
2. Hence, the drift term disappears in (SDE), and $(X, \tilde{X})$ satisfies

$$X_t = \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha)\, \mathbb{1}_{\{\tilde{X}_{s-}(\alpha) > 0\}} \mathbb{1}_{\left\{u \leqslant \frac{K(X_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds, d\alpha, du)$$

$$- \int_0^t \int_0^\infty \int_0^\infty y\, \mathbb{1}_{\{y \in (0, X_{s-})\}} \mathbb{1}_{\left\{u \leqslant \frac{X_{s-}-y}{X_{s-}} F(y, X_{s-}-y)\right\}} M(ds, dy, du). \tag{3.3}$$

3. If $Q_t = \mathscr{L}(X_t)$, then $\{Q_t\}_{t \geqslant 0}$ is a solution of (1.2) with initial condition $Q_0 = \delta_0$.

First of all, we note that the assumption $K(0, 0) > 0$ has a physical meaning. Indeed, the more a particle is small, the more it moves fast, and the more it may coalesce with others (well-known examples are $K(x, y) = (x^{1/3} + y^{1/3})(x^{-1/3} + y^{-1/3})$ or $K(x, y) = (x+1)(y+1)$, see Aldous[1]).

We note that Eq. (3.3) clearly admits $(X, \tilde{X}) = (0, 0)$ as a trivial solution. Thus, uniqueness does not hold for (3.3). We are not able to prove uniqueness (in law) for (SDE), but it might hold: $(X, \tilde{X}) = (0, 0)$ is not a solution of (SDE).

As a third remark, note that intuitively, when $F = 0$, the solutions $\{Q_t\}_{t \geqslant 0}$ to Eq. (1.2) satisfy the condition that for each t, Q_t has its support in the closure of $\mathscr{H}=\{x_1+\cdots+x_n; n\in\mathbb{N}, x_i\in\operatorname{supp} Q_0\}$. This is not the case here, since $\mathscr{H}=\{0\}$.

We finally would like to study the particular case where $K=1$ and $F=0$. In such a case, Deaconu and Tanré[6] studied the large time behaviour of solutions to (1.1). They showed that any solution $c(t,x)$ to (1.1), properly rescaled, converges to $\bar{c}(t,x)=\frac{4}{t^2}e^{-2x/t}$.

Proposition 3.7. Assume that $K=1$, that $F=0$, and that $Q_0=\delta_0$.

1. There is uniqueness for (CF). The unique solution $\{Q_t\}_{t\geqslant 0}$ is given, for all $t>0$, by

$$Q_t(dx)=\frac{4x}{t^2}e^{-2x/t}\,dx.$$

2. Consider a solution $(X,\tilde{X})$ to (SDE). Then there exists a Poisson measure $\nu(ds,dx)$ on $[0,\infty)\times(0,\infty)$ with intensity $ds\,\frac{4}{s^2}e^{-2x/s}\,dx$ such that for all t,

$$X_t=\int_0^t\int_0^\infty x\nu(ds,dx).$$

In particular, there is uniqueness in law for (SDE).

Note more generally, that Eq. (3.3) arises when studying the long time behaviour of the process X, see ref. 6.

3.4. Producing Some Dust

We first consider the pure fragmentation case.

Proposition 3.8. Assume (A), and $K=0$. Assume that for some $\rho>0$, some $\gamma\in(0,1)$, $\psi(x)\geqslant\rho x^\gamma$. Consider a solution $(X,\tilde{X})$ to (SDE), and set

$$\tau_0=\inf\{t>0; X_t=0\}.$$

1. Then $E(\tau_0)<\infty$. Furthermore, for all $t\in(0,\infty)$, $X_{\tau_0+t}=0$ a.s., $P(\tau_0<t)>0$. If we assume that $P(X_0>0)>0$, then $P(\tau_0>t)>0$ for every $t\geqslant 0$.

2. Consider now the solution $Q_t=\mathscr{L}(X_t)$ to (CF). This solution is also a solution to (1.2), and $Q_t(\{0\})$ is positive as soon as $t>0$, and is an increasing function of t with limit 1.

We now extend some of the previous results to the case where the coagulation kernel is much weaker than the fragmentation for the small particles.

Corollary 3.9. Assume (A). Assume that for some $\rho > 0$, some $\gamma \in (0, 1)$, $\psi(x) \geqslant \rho x^{\gamma}$, and that $K(x, y) \leqslant Cxy$ for some $C \geqslant 0$. Consider a solution $(X, \tilde{X})$ to (SDE), and set

$$\tau_0 = \inf\{t > 0; X_t = 0\}.$$

1. Then for all $t \in (0, \infty)$, $P(\tau_0 < t) > 0$. Furthermore, $X_{\tau_0 + t} = 0$ on the set where $\tau_0 < \infty$. If we assume that $P(X_0 > 0) > 0$, then $P(\tau_0 > t) > 0$ for every $t \geqslant 0$.

Note that since $K(0, y) = 0$ for all y, the drift term in (SDE) does not appear.

2. Consider now the solution $Q_t = \mathscr{L}(X_t)$ to (CF). This solution is also a solution to (1.2), and $Q_t(\{0\})$ is positive as soon as $t > 0$, and is an increasing function of t.

The following remark shows that the condition on ψ is justified, in the sense that converse results hold.

Remark 3.10. Assume (A), that $P(X_0 = 0) = 0$, and the hypothesis of Remark 3.3. Consider a solution $(X, \tilde{X})$ to (SDE). If $\psi(x) \leqslant C(x + x^p)$, then X never reaches 0. This means that $P(X_t = 0) = 0$ for all t.

We finally study a particular case where the drift term of (SDE) contributes to the dynamics, since fragmentation leads to the creation of dust, while $K(0, x) \neq 0$ except if $x = 0$.

Proposition 3.11. Assume (A), that $K(x, y) \leqslant x^{\gamma} + y^{\gamma}$ and that $\psi(x) \geqslant \rho x^{\gamma}$, for some $\gamma \in (0, 1)$. Consider a solution $(X, \tilde{X})$ to (SDE), and denote by $Q_t = \mathscr{L}(X_t)$ the corresponding solution to (CF). Then if $\rho > 4/(1-\gamma)$,

$$\limsup_{t \to \infty} E\left[\frac{1}{t}\int_0^t \mathbb{1}_{\{X_s > 0\}}\, ds\right] = \limsup_{t \to \infty} \frac{1}{t}\int_0^t Q_s((0, \infty))\, ds < 1. \tag{3.4}$$

In particular, the drift term does not vanish identically in (SDE).

Note that in such a case, the mass X_t of the "typical" particle may be 0 at some instant, and positive after this instant. Thus (3.4) gives more information than the inequality $E[\tau_0] < \infty$.

3.5. On the Regularity of Q_t

We finally show that under some conditions, the solution $Q_t(dx)$ of (CF) has a density $f(t, x)$ with respect to the Lebesgue measure (or eventually to $dx + \delta_0(dx)$), even if Q_0 is singular. This is made possible by the infinite fragmentation rate.

We only study a particular case, because the method we use, based on the stochastic calculus of variations for jump processes, is quite restrictive.

Proposition 3.12. Assume (A), and that $K(x, y) \leqslant Cxy$. Assume also that the conditions of Remark 3.3 are fulfilled. Assume that β is C^1 on $(0, 1)$, that $\int_0^1 \beta(\theta)\, d\theta = \infty$, and that the map $x \mapsto x\alpha(x)$ is non-increasing from $(0, \infty)$ into itself. Consider a solution $(X, \tilde{X})$ to (SDE), and denote by $Q_t = \mathscr{L}(X_t) = \mathscr{L}_\alpha(\tilde{X}_t)$ the corresponding solution to (CF).

1. Then as soon as $t > 0$, Q_t has a density with respect to $dx + \delta_0(dx)$.
2. In the very particular case where $x \mapsto x\alpha(x)$ is bounded, and where $Q_0(\{0\}) = 0$, Q_t has a density with respect to dx, as soon as $t > 0$.

Note that in 1, we could not obtain absolute continuity with respect to the Lebesgue measure, with regard to Proposition 3.8 and the equality $x\alpha(x) = \psi(x)/x$.

One may hope however that the assumptions are much too strong. Indeed, it seems reasonable to conjecture that for $\{Q_t\}_{t \geqslant 0}$ a solution to (CF) and $t > 0$:

(i) Under Assumption (A), Q_t has a density with respect to $dx + \delta_0(dx)$ as soon as $\int_0^x F(y, x-y)\, dy = \infty$ for all $x > 0$. Here the regularization comes from the "infinite" rate of "continuous" fragmentation.

(ii) Under Assumption (A) and $\int_0^x F(y, x-y)\, dy \leqslant C(1 + x^{p+1})$, if $Q_0(\{0\}) > 0$ and $K(x, y) > 0$ on $[0, \infty)^2$, Q_t has a density with respect to dx. Here the regularization comes from the coalescence of infinitely small particles on X.

We are, alas, not able to obtain such results.

4. PROOFS

First of all, we omit to expose a proof of Proposition 3.1, since it is now a rather standard result. Two schemes may be proposed: either use

Laurençot's solution to (CF),[14] to build a solution to (SDE), or adapt the proof of ref. 7. Thus only the stability result is really new.

Proof of Theorem 3.2. First of all, we prove 1. It suffices to check that $\{Q^n\}_n$ satisfies the Aldous criterion for tightness, see Jacod and Shiryaev,[12] p. 320. We will check the two points below, which suffices:

(i) For all $T<\infty$,

$$\sup_n E(\sup_{[0,T]} (X_t^n)^{p+1}) \leqslant C_T. \tag{4.1}$$

(ii) Denote, for $T<\infty$ and $\delta>0$, by $\mathscr{A}(T,\delta)$ the set of couples of stopping times (S,S') satisfying a.s. $0\leqslant S\leqslant S'\leqslant S+\delta\leqslant T$. Then

$$\sup_n \sup_{(S,S')\in\mathscr{A}(T,\delta)} E[|X_{S'}^n - X_S^n|] \leqslant C_T\delta.$$

An easy computation using Assumption (A) (1 and 3), the fact that $\mathscr{L}(X^n)=\mathscr{L}_\alpha(\tilde{X}^n)$, and that fragmentation makes X decrease, shows that

$$\begin{aligned}
E[\sup_{[0,t]} (X_s^n)^{p+1}] &\leqslant E[(X_0\vee(1/n))^{p+1}] \\
&\quad + \int_0^t EE_\alpha\left[\frac{K(X_s^n,\tilde{X}_s^n)}{\tilde{X}_s^n}([X_s^n+\tilde{X}_s^n]^{p+1}-[X_s^n]^{p+1})\right]ds \\
&\leqslant C_p + C_p\int_0^t EE_\alpha[K(X_s^n,\tilde{X}_s^n)(1+(X_s^n)^p+(\tilde{X}_s^n)^p)]\,ds \\
&\leqslant C_p + C_p\int_0^t E[1+[X_s^n]^{p+1}]\,ds.
\end{aligned}$$

Using the Gronwall Lemma allows to conclude that (i) holds.

Consider now (S,S') in $\mathscr{A}(T,\delta)$. Using (A) and (4.1), we obtain

$$\begin{aligned}
E[|X_{S'}^n - X_S^n|] &\leqslant E\left[\int_S^{S+\delta} E_\alpha(K(X_s^n,\tilde{X}_s^n))\,ds\right] + E\left[\int_S^{S+\delta}\psi(X_s^n)\,ds\right] \\
&\leqslant CE\left[\int_S^{S+\delta} E_\alpha(1+X_s^n+\tilde{X}_s^n)\,ds\right] + CE\left[\int_S^{S+\delta}[1+(X_s^n)^p]\,ds\right] \\
&\leqslant C_T E[\sup_{[0,T]}(1+X_s^n+[X_s^n]^p)]\times\delta \leqslant C_T\times\delta
\end{aligned}$$

and (ii) also holds.

We now present a proof of 2. We thus consider a converging-in-law subsequence of X^n which we still denote by X^n, going to some limiting point X. We set $Q^n = \mathscr{L}(X^n) = \mathscr{L}_\alpha(\tilde{X}^n)$ and by $Q = \mathscr{L}(X)$. Denote by Q^n_s and Q_s their time marginal laws at s. We will prove that Q satisfies the martingale problem associated with (SDE), which suffices, according to standard representation theorems for point processes. Thus what we have to show is that

(iii) $\mathscr{L}(X_0) = Q_0$,

(iv) for all $\phi \in C^1_b([0, \infty[)$, (we use the notations of Definition 2.1), the process

$$M^\phi_t = \phi(X_t) - \phi(X_0) - \int_0^t [\mathscr{K}_{Q_s}\phi(X_s) + \mathscr{F}\phi(X_s)]\, ds$$

is a martingale.

First, (iii) is immediate, since $\mathscr{L}(X_0)$ is the weak limit of $\mathscr{L}(X^n_0) = Q^n_0 = Q_0([0, 1/n])\, \delta_{1/n}(dx) + \mathbb{1}_{[1/n, \infty[}(x)\, Q_0(dx)$.

Next, we check (iv). We shall show that for any $0 \leqslant s_1 < \cdots < s_k \leqslant s < t$, any $g_1,..., g_k$ in $C_b([0, \infty))$, and any $\phi \in C^1_b([0, \infty))$,

$$E\left[\prod_{i=1}^k g_i(X_{s_i}) \times (\phi(X_t) - \phi(X_s) - \int_s^t [\mathscr{K}_{Q_u}\phi(X_u) + \mathscr{F}\phi(X_u)]\, du)\right] = 0. \tag{4.2}$$

What we know is that

$$E\left[\prod_{i=1}^k g_i(X^n_{s_i}) \times (\phi(X^n_t) - \phi(X^n_s) - \int_s^t [\mathscr{K}_{Q^n_u}\phi(X^n_u) + \mathscr{F}_n\phi(X^n_u)]\, du)\right] = 0 \tag{4.3}$$

where $\mathscr{F}_n$ is defined as $\mathscr{F}$ with F replaced by $F_n = F \wedge n$. First of all, the map I from $\mathbb{D}([0, \infty), \mathbb{R}_+)$ into $\mathbb{R}$ defined by $x \mapsto \prod_{i=1}^k g_i(x_{s_i}) \times (\phi(x_t) - \phi(x_s))$ is continuous at any point x which has no jumps at $s_1,..., s_k, s, t$. This is a.s. the case of X, which is quasi-left-continuous (see Jacod and Shiryaev,[12] p. 22), since it is the limit in law of a sequence of processes satisfying the Aldous criterion (see ref. 12, p. 320). Furthermore, it is clear that $I(x) \leqslant C(1 + \sup_{[0,t]} x_u)$. Using the uniform integrability of $(\sup_{[0,t]} X^n_s)$ obtained in (4.1), we deduce that, as n tends to infinity,

$$E\left[\prod_{i=1}^k g^n_i(X_{s_i}) \times (\phi(X^n_t) - \phi(X^n_s))\right] \to E\left[\prod_{i=1}^k g_i(X_{s_i}) \times (\phi(X_t) - \phi(X_s))\right]. \tag{4.4}$$

Secondly, for any q, any x, we can write $\mathscr{K}_q\phi(x)=\int_0^\infty \alpha(x,y)\,q(dy)$, where $\alpha(x,y)=K(x,y)[\phi(x+y)-\phi(x)]/y$ if $y>0$, and $\alpha(x,0)=\phi'(0)\,K(x,0)$. Hence we see that $\int_s^t \mathscr{K}_{Q_u^n}\phi(X_u^n)\,du=\langle Q^n, \int_s^t \alpha(X_u^n,.)\,du\rangle$. The map α being continuous on $[0,\infty)^2$, we deduce that the map from $[\mathbb{D}([0,\infty),\mathbb{R}_+)]^2$ into $\mathbb{R}$ defined by $(x,y)\mapsto\int_s^t \alpha(x_u,y_u)\,du$ is continuous, and bounded from above by $C\sup_{[0,t]}[1+x_u+y_u]$. Using the uniform integrability of $(\sup_{[0,t]} X_s^n)$ obtained in (4.1), one easily deduces that, as n tends to infinity,

$$E\left[\prod_{i=1}^k g_i(X_{s_i}^n)\times\int_s^t \mathscr{K}_{Q_u^n}\phi(X_u^n)\,du\right]\to E\left[\prod_{i=1}^k g_i(X_{s_i})\times\int_s^t \mathscr{K}_{Q_u}\phi(X_u)\,du\right]. \tag{4.5}$$

We finally have to check that as n increases to infinity,

$$\begin{aligned}A_n&=E\left[\prod_{i=1}^k g_i(X_{s_i}^n)\times\int_s^t \mathscr{F}_n\phi(X_u^n)\,du\right]\\&\to A=E\left[\prod_{i=1}^k g_i(X_{s_i})\times\int_s^t \mathscr{F}\phi(X_u)\,du\right].\end{aligned} \tag{4.6}$$

We denote by B_n the same expression as A_n with $\mathscr{F}_n$ replaced by $\mathscr{F}$. Then, it is clear that $|B_n-A|$ tends to 0, since the map $x\mapsto\mathscr{F}\phi(x)$ is continuous (thanks to (A)-2) and bounded from above by $C\psi(x)\leqslant C(1+x^p)$, and thanks to the uniform integrability of $\sup_{[0,t]}|X^n|^p$. On the other hand, it is clear that

$$\begin{aligned}|B_n-A_n|&\leqslant C\int_s^t E\left[\frac{1}{X_u^n}\int_0^{X_u^n} y(X_u^n-y)\,F(y,X_u^n-y)\,\mathbb{1}_{\{F(y,X_u^n-y)\geqslant n\}}\,dy\right]du\\&=C\int_s^t \Delta_u^n\,du\end{aligned}$$

where the last equality stands for a definition. Since for all u, $\Delta_u^n\leqslant E[\psi(X_u^n)]\leqslant CE[1+(X_u^n)^p]\leqslant C_T$ thanks to (4.1), one may use the Lebesgue theorem, and it remains to show that for each u, Δ_u^n tends to 0. For each $\varepsilon>0$, we use the decomposition

$$\begin{aligned}\Delta_u^n\leqslant&\sup_{x\in[\varepsilon,1/\varepsilon]}\frac{1}{x}\int_0^x y(x-y)\,F(y,x-y)\,\mathbb{1}_{\{F(y,x-y)\geqslant n\}}\,dy\\&+E[\mathbb{1}_{\{X_u^n<\varepsilon\}}\psi(X_u^n)]+E[\mathbb{1}_{\{X_u^n>1/\varepsilon\}}\psi(X_u^n)].\end{aligned}$$

Thanks to (A), the first term on the right-hand side goes to 0 as n tends to infinity, for each $\varepsilon > 0$ fixed. The second term is smaller than $\sup_{[0,\varepsilon]} \psi(x)$, which tends to 0, uniformly in n as ε tends to 0, since Assumption (A) ensures that $\psi(0) = 0$ and ψ is continuous. Finally, the third term also tends to 0 uniformly in n as ε tends to 0, since $\psi(x) \leqslant C(1+x^p)$, and from the uniform integrability of $(X_u^n)^p$ obtained in (4.1). Hence $|B_n - A_n|$ tends to 0.

Associating (4.4), (4.5), and (4.6) allows to go to the limit in (4.3) and to establish (4.2). ∎

We now give the way we obtain the specific version (3.1) of (SDE) in the case of fragmentation kernels described in Remark 3.3.

Proof of Remark 3.3. We make the "substitution" $\theta = y/X_{s-}$ in the fragmentation term of (SDE). We set

$$A_t = \int_0^t \int_0^\infty \int_0^\infty y \mathbb{1}_{\{y \in]0, X_{s-}[\}} \mathbb{1}_{\left\{u \leqslant \frac{X_{s-}-y}{X_{s-}} F(y, X_{s-}-y)\right\}} M(ds, dy, du).$$

Then, due to the specific form of F,

$$A_t = \int_0^t \int_0^\infty \int_0^\infty \frac{y}{X_{s-}} X_{s-} \mathbb{1}_{\left\{\frac{y}{X_{s-}} \in]0,1[\right\}} \mathbb{1}_{\left\{u \leqslant \left(1-\frac{y}{X_{s-}}\right) \alpha(X_{s-}) \beta\left(\frac{y}{X_{s-}}\right)\right\}} M(ds, dy, du).$$

$(T_i, Y_i, U_i)_{i \geqslant 1}$ stand for the points in the support of M. In other words, $M(ds, dy, du) = \sum_{i \geqslant 1} \delta_{(T_i, Y_i, U_i)}(ds, dy, du)$. Consider the point measure $\mu(ds, d\theta, dv) = \sum_{i \geqslant 1} \delta_{(T_i, \theta_i, V_i)}(ds, d\theta, dv)$, where

$$\theta_i = Y_i / X_{T_i-}; \qquad V_i = U_i X_{T_i-} / [(1-\theta_i) \beta(\theta_i)].$$

Then one easily checks that the compensator of μ is deterministic (hence μ is Poisson) and is given by $ds(1-\theta) \beta(\theta) \, d\theta \, dv$. It is furthermore straightforward that

$$A_t = \int_0^t \int_0^1 \int_0^\infty \theta X_{s-} \mathbb{1}_{\{v \leqslant X_{s-} \alpha(X_{s-})\}} \mu(ds, d\theta, du).$$

This concludes the proof. ∎

We now study the "finite case."

Proof of Proposition 3.4. Since $F = 0$ and $X_0 \geqslant 1$ a.s., it is clear that for all t, $X_t > 0$ a.s. Thus $(X, \tilde{X})$ simply satisfies the equation

$$X_t = X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\{\tilde{X}_{s-}(\alpha) > 0\}} \mathbb{1}_{\left\{u \leqslant \frac{K(X_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds, d\alpha, du).$$

We now show recursively on n that a.s., for all t, X_t does not belong to $]n, n+1[$.

Since $\tilde{X}$ is a nonnegative process, it is clear that X is an non-decreasing process. In particular, we deduce that for all t, $X_t \geqslant X_0 \geqslant 1$ a.s. In particular, we see that a.s., X_t does not belong to $]0, 1[$.

Let now $n \geqslant 1$ be fixed, and assume that a.s., for all t, all $1 \leqslant k \leqslant n$, $X_t \notin]k-1, k[$ (and thus $\tilde{X}_t \notin]k-1, k[$ $d\alpha$-a.s.). Assume that for some $t \geqslant 0$, $X_t \in]n, n+1[$ with positive probability. Since X is a.s. nondecreasing and starts from an integer $X_0 \geqslant 1$, we deduce that with positive probability, X has jumps of size in $\bigcup_{k=1}^{n}]k-1, k[$. But the jumps of X are of the form $\tilde{X}_u(\alpha)$, and are thus a.s. never in $\bigcup_{k=1}^{n}]k-1, k[$. Hence, with probability one, for all $t \geqslant 0$, $X_t \notin]n, n+1[$. This concludes the induction. ∎

We now check that if the initial condition has a density and if the fragmentation is finite, then the solution has a density.

Proof of Proposition 3.5. First of all, a simple computation, using (3.2), point (ii) in the definition of (SDE) (see Definition 2.3) and the fact that $E(X_0^{-1}) < \infty$ shows that for all $T > 0$,

$$E[\sup_{[0,T]} (X_t)^{-1}] \leqslant C_T. \tag{4.7}$$

We now consider, for all $a > 0$, the set

$$\mathscr{A}_a = \left\{ A \text{ Borelian subset of } [0, a];\ \int_A dx = 0 \right\}.$$

Our aim is to prove that for each $a > 0$ fixed, the function $\phi_a(t) = \sup_{A \in \mathscr{A}_a} P(X_t \in A)$ vanishes identically. This will show that for all $t \geqslant 0$, $\mathscr{L}(X_t) \ll dx$. Let A belong to $\mathscr{A}_a$. We know that $P(X_0 \in A) = 0$. Then, using the negativity of certain terms and $\mathscr{L}(X_0) \ll dx$, we obtain

$$\begin{aligned}
P(X_t \in A) &= P(X_0 \in A) + \int_0^t EE_\alpha \left[(\mathbb{1}_A(X_s + \tilde{X}_s) - \mathbb{1}_A(X_s)) \frac{K(X_s, \tilde{X}_s)}{\tilde{X}_s} \right] ds \\
&\quad + \int_0^t E\left[\int_0^{X_s} (\mathbb{1}_A(X_s - y) - \mathbb{1}_A(X_s)) \frac{X_s - y}{X_s} F(y, X_s - y)\, dy \right] ds \\
&\leqslant \int_0^t EE_\alpha \left[\mathbb{1}_A(X_s + \tilde{X}_s) \frac{K(X_s, \tilde{X}_s)}{\tilde{X}_s} \right] ds \\
&\quad + \int_0^t E\left[\int_0^{X_s} \mathbb{1}_A(X_s - y)\, F(y, X_s - y)\, dy \right] ds.
\end{aligned}$$

First note that thanks to (3.2), a.s., $\int_0^{X_s} \mathbb{1}_A(X_s - y)\, F(y, X_s - y)\, dy = 0$, since A is Lebesgue-null. Thus, using (A), and denoting by Q_s the common law of X_s and $\tilde{X}_s$,

$$P(X_t \in A) \leqslant \int_0^t EE_\alpha \left[\mathbb{1}_A(X_s + \tilde{X}_s)\, \mathbb{1}_{\{X_s \leqslant a\}} \left(\frac{1+a}{\tilde{X}_s} + 1 \right) \right] ds$$

$$\leqslant \int_0^t ds \int_0^a Q_s(dy)\, P[X_s \in A - y] \left(\frac{1+a}{y} + 1 \right).$$

But for each $y \in [0, a]$, $A - y$ belongs to $\mathscr{A}_a$. Using furthermore (4.7), we obtain

$$P(X_t \in A) \leqslant C_{T,a} \int_0^t \phi_a(s)\, ds.$$

Taking the supremum over all $A \in \mathscr{A}_a$ in the left-hand side member, and applying the Gronwall Lemma allow to conclude. ∎

We now study what happens when Q_0 is a Dirac mass at 0: the system is initially composed of a "cloud of dust."

Proof of Proposition 3.6. Note that points 2 and 3 are straightforward consequences of 1. We thus only check point 1. Denote by $J_t^F = \sum_{s \leqslant t} \mathbb{1}_{\Delta X_s < 0}$ the number of jumps of X on $[0, t]$ due to the fragmentation. Then for all $T < \infty$,

$$E[J_T] = E\left[\int_0^T \int_0^{X_s} \frac{X_s - y}{X_s} F(y, X_s - y)\, dy\, ds \right]$$

$$\leqslant C_T E[\sup_{[0,T]} (1 + X_t^{p+1})] < \infty$$

by the assumption on F and by condition (ii) of Definition 2.3.

Denote by Ω_t the event $\{X_t = 0\}$. We first prove that Ω_t does not increase as t increases. To this aim, let t be positive and ω lie in Ω_t^c. Then for all $r \geqslant t$, $X_r(\omega) > 0$. Indeed, from $J_r^F - J_t^F < \infty$ a.s., we know that the fragmentation integral in (SDE) has finitely many jumps. We denote by $T_1^F, \ldots, T_n^F$ the instants of these jumps. Between two instants of fragmentation, we know that $X(\omega)$ does not decrease, while it is also clear that at each instant T_i^F of fragmentation, $-\Delta X_{T_i^F} \leqslant X_{T_i-}$. Hence $X_r(\omega) > 0$.

We now deduce that it is not possible that for some $\delta > 0$, $P(X_\delta = 0) > 0$. Assume the converse, and consider $\omega \in \Omega_\delta$ with $S_1^F(\omega) > 0$,

S_1^F denoting the first instant of negative jump of X. Then it is easily deduced from equation (SDE) that for $t \in]0, S_1^F(\omega) \wedge \delta[$,

$$X_t \geqslant \int_0^t K(0, X_s)\, P(X_s = 0)\, ds \geqslant P(\Omega_\delta)\, K(0, 0)\, t > 0.$$

But in such a case, we have $X_t(\omega) > 0$, and thus that $\omega \notin \Omega_t$. Since Ω_t is non increasing, we deduce that $\omega \notin \Omega_\delta$. This is a contradiction. ∎

We now solve explicitely (SDE) in the pure coagulation case, when $K = 1$ and $X_0 = 0$.

Proof of Proposition 3.7. We fragment the proof in several steps.

Step 1. First, an easy computation shows that $Q_t(dx) = \frac{4x}{t^2}\, e^{-2x/t}\, dx$ satisfies (CF) in the sense of Definition 2.1.

Step 2. We now prove the uniqueness for (CF). We thus consider two solutions $\{Q_t\}_{t \geqslant 0}$ and $\{R_t\}_{t \geqslant 0}$ with $Q_0 = R_0 = \delta_0$, and we introduce the set $C^\infty_{b,1}$ of C^∞ functions ϕ on $[0, \infty[$, bounded by 1 with all their derivatives. For such a ϕ, we define $\alpha^\phi(x, y) = [\phi(x+y) - \phi(x)]/y$ if $y > 0$, and $\alpha^\phi(x, 0) = \phi'(x)$. Then we have

$$\begin{aligned} |\langle Q_t - R_t, \phi \rangle| &\leqslant \int_0^t |\langle Q_s \otimes Q_s - R_s \otimes R_s, \alpha^\phi \rangle|\, ds \\ &\leqslant \int_0^t |\langle Q_s - R_s, \beta^{\phi, Q_s} \rangle| + |\langle R_s - Q_s, \gamma^{\phi, Q_s} \rangle|\, ds, \end{aligned}$$

where $\beta^{\phi, q}(y) = \int q(dy)\, \alpha^\phi(x, y)$ and $\gamma^{\phi, q}(x) = \int q(dy)\, \alpha^\phi(x, y)$. One may easily check that for any $\phi \in C^\infty_{b,1}$, any probability measure $q \in \mathscr{P}([0, \infty[)$, the maps $\beta^{\phi, q}$ and $\gamma^{\phi, q}$ also belong to $C^\infty_{b,1}$. We thus obtain, using the Gronwall Lemma, that for all $t \geqslant 0$, $\sup_{\phi \in C^\infty_{b,1}} |\langle Q_t - R_t, \phi \rangle| = 0$.

We deduce in particular that for all $t \geqslant 0$, all $\xi \in [-1, 1]$, $\langle Q_t(dx), e^{i\xi x} \rangle = \langle R_t(dx), e^{i\xi x} \rangle$.

Furthermore, one easily checks that for all $t \geqslant 0$, there exists $\gamma_t > 0$ such that $\langle Q_t(dx) + R_t(dx), e^{\gamma_t x} \rangle < \infty$. Hence, for each $t \geqslant 0$, the applications $\phi_{Q_t}(z) = \langle Q_t(dx), e^{zx} \rangle$ and $\phi_{R_t}(z) = \langle R_t(dx), e^{zx} \rangle$ are holomorphic on $\{\mathrm{Re}\, z < \alpha_t\}$.

Since they coincide on $\{z = i\xi, \xi \in [-1, 1]\}$, they coincide on $\{\mathrm{Re}\, z < \alpha_t\}$. We deduce in particular that for all $\lambda \geqslant 0$, $\langle Q_t(dx), e^{-\lambda x} \rangle = \langle R_t(dx), e^{-\lambda x} \rangle$. Hence for any $t \geqslant 0$, R_t and Q_t have the same Laplace transform, and thus are equal.

Step 3. We now prove point 2. Consider a solution $(X, \tilde{X})$ to (SDE). We know that $\{\mathscr{L}_\alpha(\tilde{X}_t)\}_{t\geqslant 0}$ satisfies (CF). Hence, the two previous steps yield that for each $t>0$, $\mathscr{L}_\alpha(\tilde{X}_t)(dx)=\frac{4x}{t^2}\,e^{-2x/t}\,dx$. Making the substitution $x=\tilde{X}_s(\alpha)$ in equation (SDE) implies (see the proof of Remark 3.3 for the rigorous arguments of such a substitution)

$$X_t=\int_0^t\int_0^1\int_0^\infty \tilde{X}_s(\alpha)\,\mathbb{1}_{\{u\leqslant 1/\tilde{X}_s(\alpha)\}}N(ds, d\alpha, du)$$

$$=\int_0^t\int_0^\infty\int_0^\infty x\mathbb{1}_{\{u\leqslant 1/x\}}\Gamma(ds, dx, du)$$

where $\Gamma(ds, dx, du)$ is a Poisson measure with intensity $ds\,\frac{4x}{t^2}\,e^{-2x/t}\,dx\,du$. Then, the Poisson measure $\nu(ds, dx)=\int_0^\infty \mathbb{1}_{u\leqslant x}\Gamma(ds, dx, du)$ has the intensity $ds\,\frac{4}{t^2}\,e^{-2x/t}\,dx$, and

$$X_t=\int_0^t\int_0^\infty x\nu(ds, dx).$$

This concludes the proof. ∎

We now prove that if the fragmentation is sufficiently explosive, and if $K=0$, then X_t reaches 0.

Proof of Proposition 3.8. One more time, point 2 is a direct consequence of point 1. To prove point 1, we proceed in four steps.

Step 1. We first show that $E[\tau_0]<\infty$. We compute $E[X_t^{1-\gamma}]$, where γ is defined in the statement, and we make τ_0 appear. Since the map $x^{1-\gamma}$ is not of class C_b^1, we have to introduce, for each $\varepsilon>0$, an increasing C_b^1 function $\zeta^\varepsilon(x)$ such that $\zeta^\varepsilon(x)=x^{1-\gamma}$ for all $x\geqslant\varepsilon$.

$$\begin{aligned}E[\zeta^\varepsilon(X_t)]&=E[\zeta^\varepsilon(X_0)]-\int_0^t E\left[\frac{1}{X_s}\int_0^{X_s}\{\zeta^\varepsilon(X_s)-\zeta^\varepsilon(X_s-y)\}\right.\\&\qquad\left.\times(X_s-y)\,F(y, X_s-y)\,dy\right]ds\\&\leqslant E[\zeta^\varepsilon(X_0)]-\int_0^t E\left[\mathbb{1}_{\{X_s\geqslant 2\varepsilon\}}\frac{1}{X_s}\int_\varepsilon^{X_s}\{(X_s)^{1-\gamma}-(X_s-y)^{1-\gamma}\}\right.\\&\qquad\left.\times(X_s-y)\,F(y, X_s-y)\,dy\right]ds.\end{aligned}$$

But one may prove, using the symmetry of F, that for all $x > 0$,

$$\frac{1}{x}\int_{x/2}^{x}\{x^{1-\gamma}-(x-y)^{1-\gamma}\}(x-y)\,F(y,x-y)\,dy \geqslant \frac{(1-\gamma)}{2}\frac{\psi(x)}{x^{\gamma}}.$$

Using the assumption on ψ and the fact the $X_t \geqslant 0$ a.s, we obtain

$$E\left[\int_0^t \mathbb{1}_{\{X_s \geqslant 2\varepsilon\}}\,ds\right] \leqslant \frac{2}{\rho(1-\gamma)}\,E[\zeta^{\varepsilon}(X_0)].$$

Making ε tend to 0 yields that

$$E\left[\int_0^t \mathbb{1}_{\{X_s > 0\}}\,ds\right] \leqslant \frac{2}{\rho(1-\gamma)}\,E[X_0^{1-\gamma}].$$

Making t tend to infinity shows finally that

$$E[\tau_0] \leqslant \frac{2}{\rho(1-\gamma)}\,E[X_0^{1-\gamma}] < \infty. \tag{4.8}$$

Step 2. The assertion $X_{\tau_0+t} = 0$ for all $t \geqslant 0$ is obvious, since in the pure fragmentation case, $s \mapsto X_s$ does not increase a.s.

Step 3. We now prove that for any $t > 0$, $P[\tau_0 \leqslant t] > 0$. Assume the converse. Then there exists t_0 such that for all $t \leqslant t_0$, $X_t > 0$ a.s. We will deduce the following points:

(i) for all $\varepsilon > 0$, $P(X_{t_0/2} < \varepsilon) > 0$,

(ii) denote by X_t^x the process conditioned by $X_0 = x$, and by τ_0^x the corresponding hitting time. There exists a sequence x_n going to 0 as n tends to infinity such that for any n, $\tau_0^{x_n} \geqslant t_0/2$ a.s.,

(iii) $E(\tau_0^{x_n})$ goes to 0 when n tends to infinity.

Points (ii) and (iii) are in contradiction.
We first check (i).

$$P(X_{t_0/2} < \varepsilon) \geqslant P[U^{\varepsilon}_{t_0/2} \geqslant 1]$$

where

$$U_t^{\varepsilon} = \int_0^t\int_0^{\infty}\int_0^{\infty} \mathbb{1}_{\{y\in\,]X_{s-}-\varepsilon,\, X_{s-}[\}}\mathbb{1}_{\left\{u \leqslant \frac{X_{s-}-y}{X_{s-}}F(y,\,X_{s-}-y)\right\}}\,M(ds,dy,du).$$

But a simple computation shows that the integer-valued random variable $U^{\varepsilon}_{t_0/2}$ has a positive expectation. Hence, $P[U^{\varepsilon}_{t_0/2} \geqslant 1] > 0$, which concludes the proof of (i).

(ii) Note that since we consider the pure fragmentation case, X^x_t is an homogeneous Markov process. From $\tau_0 \geqslant t_0$ a.s. and the Markov property, we deduce that $P[\tau_0^{X_{t_0/2}} \geqslant t_0/2] = P[\tau_0 \geqslant t_0] = 1$. Thus,

$$\int_0^\infty P[\tau_0^x \geqslant t_0/2]\, Q_{t_0/2}(dx) = 1 \tag{4.9}$$

where $Q_{t_0/2} = \mathscr{L}(X_{t_0/2})$. We deduce that for $Q_{t_0/2}$-almost all x, $P[\tau_0^x \geqslant t_0/2] = 1$. Using (i) allows to conlude.

Finally, (iii) follows from (4.8), which yields that $E[\tau_0^{x_n}] \leqslant C x_n^{1-\gamma}$.

Step 4. We finally check that for any fixed $t > 0$, $P(\tau_0 \geqslant t) > 0$. First of all, we consider $0 < x_1 < x_2$ such that $P(X_0 \in [x_1, x_2]) > 0$. Then we consider, for a positive integer n to be chosen later,

$$V^n_t = \int_0^t \int_0^\infty \int_n^\infty y \mathbb{1}_{\{y \in]0, X_{s-}[\}} \mathbb{1}_{\left\{u \leqslant \frac{X_{s-} - y}{X_{s-}} F(y, X_{s-} - y)\right\}} M(ds, dy, du),$$

$$W^n_t = \int_0^t \int_0^\infty \int_0^n \mathbb{1}_{\{y \in]0, X_{s-}[\}} \mathbb{1}_{\left\{u \leqslant \frac{X_{s-} - y}{X_{s-}} F(y, X_{s-} - y)\right\}} M(ds, dy, du).$$

Then it is clear that for any n,

$$\begin{aligned} P(\tau_0 \geqslant t) &\geqslant P(X_0 \in [x_1, x_2], X_t \geqslant x_1/2) \\ &\geqslant P(X_0 \in [x_1, x_2], V^n_t < x_1/2, W^n_t = 0). \end{aligned}$$

Using the fact that X a.s. does not increase, we obtain that

$$W^n_t \leqslant \int_0^t \int_0^{X_0} \int_0^n M(ds, dy, du) = C^n_t$$

where the equality stands for a definition. Conditionally to X_0, C^n_t follows a Poisson distribution with parameter $tX_0 n$. In particular, for any n, any $x_0 \in [x_1, x_2]$, $P(C^n_t = 0 \mid X_0 = x_0) \geqslant \exp(-ntx_2)$.

On the other hand, an easy computation using the independance of the Poisson measures $M|_{[0,\infty[\times [0,\infty[\times [0,n[}$ and $M|_{[0,\infty[\times [0,\infty[\times [n,\infty[}$, shows that for any $x_0 \in [x_1, x_2]$,

$$E[V^n_t \mid C^n_t = 0, X_0 = x_0] \leqslant \int_0^t E[\psi_n(X_s) \mid C^n_t = 0, X_0 = x_0]\, ds$$

where $\psi_n(x) = x^{-1} \int_0^x y(x-y) F(y, x-y) \mathbb{1}_{\{F(y,x-y) \geqslant n\}} dy$. Using now (A), we have that for any $0 < \varepsilon < x_2$, $\zeta_n(\varepsilon, x_2) = \sup_{[\varepsilon, x_2]} \psi_n(x)$ tends to 0 as n tends to infinity. Furthermore, it is clear that $\psi_n \leqslant \psi$, and thanks to (A) again, we know that $\sup_{[0,\varepsilon]} \psi(x)$ tends to 0 with ε. Using finally the fact that X is a.s. non increasing, we obtain that for any $\varepsilon > 0$,

$$\inf_{x_0 \in [x_1, x_2]} E[V_t^n \mid C_t^n = 0, X_0 = x_0] \leqslant t\zeta_n(\varepsilon, x_2) + t \sup_{[0,\varepsilon]} \psi(x). \tag{4.10}$$

This clearly implies that the left-hand side of (4.10) tends to 0. Thus,

$$\lim_n \inf_{x_0 \in [x_1, x_2]} P[V_t^n \leqslant x_1/2 \mid C_t^n = 0, X_0 = x_0] = 1.$$

We choose finally n in such a way that

$$\inf_{x_0 \in [x_1, x_2]} P[V_t^n \leqslant x_1/2 \mid C_t^n = 0, X_0 = x_0] > 0.$$

We finally obtain

$$P(\tau_0 \geqslant t) \geqslant \int_{x_1}^{x_2} Q_0(dx_0)\, P[V_t^n \leqslant x_1/2 \mid C_t^n = 0, X_0 = x_0]\, P(C_t^n = 0 \mid X_0 = x_0)$$

which is a positive quantity. ∎

We now extend some of the previous results to the case of weak coagulation.

Proof of Corollary 3.9. One more time, point 2 is a straightforward consequence of point 1. To prove point 1, we use the previous proposition. First of all denote by X^F the pure fragmentation process associated with X_0 and M. In other words, X^F satisfies the same equation as X where K is replaced by 0.

Consider also the first instant T_1^K of positive jump of X. Then, since $K(x, y) \leqslant Cxy$ and since X decreases until T_1^K, one easily checks that a.s., $T_1^K \geqslant S$, where

$$S = \inf\left\{ t > 0;\ \int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\{u \leqslant CX_0\}} N(ds, d\alpha, du) \geqslant 1 \right\}.$$

Then we note that conditionally to X_0, S follows an exponential distribution of parameter CX_0, and is independent of X^F.

But on the event $\{S > t\}$, $X_s = X_s^F$ for all $s \leqslant t$. We denote by τ_0^F the first instant where X^F reaches 0. We obtain

$$\begin{aligned} P(\tau_0 < t) &\geqslant P(S > t, \tau_0 < t) \\ &= P(S > t, \tau_0^F < t) \\ &= E[P(S > t \mid X_0) \times P(\tau_0^F < t \mid X_0)] \\ &= E[e^{-CX_0 t} P(\tau_0^F < t \mid X_0)] \\ &= E[e^{-CX_0 t} \mathbb{1}_{\{\tau_0^F < t\}}]. \end{aligned}$$

This quantity is positive, since $X_0 < \infty$ a.s., and since $P(\tau_0^F < t) > 0$ according to Proposition 3.8.

The same argument shows that for any $t > 0$, $P(\tau_0 > t) \geqslant P(S \geqslant t, \tau_0 > t) > 0$. Finally, since $K(0, x) = 0$ for all x, it is obvious that $X_{\tau_0 + t} = 0$ for all $t > 0$ on the event $\{\tau_0 < \infty\}$. ∎

We now check that the previous condition on ψ is justified.

Proof of Remark 3.10. Under these assumptions, one may use Remark 3.3. Hence it is clear that

$$X_t \geqslant X_0 - \int_0^t \int_0^1 \int_0^\infty \theta X_{s-} \mathbb{1}_{\{u \leqslant X_{s-} \alpha(X_{s-})\}} \mu(ds, d\theta, du).$$

We deduce, using a Gronwall type formula for point measures (see the appendix of ref. 8 for the proof of a very similar result) that

$$X_t \geqslant X_0 \exp(-U_t)$$

where

$$U_t = \int_0^t \int_0^1 \int_0^\infty [-\ln(1-\theta)] \mathbb{1}_{\{u \leqslant X_{s-} \alpha(X_{s-})\}} \mu(ds, d\theta, du).$$

We just have to check that $U_t < \infty$ a.s. for any t. First note that by assumption, $x\alpha(x) \leqslant C(1 + x^{p-1})$. Hence, $U_t \leqslant V(t, M)$, where $M = \sup_{[0,t]} (C(1 + X_s^{p-1}))$ and

$$V(t, x) = \int_0^t \int_0^1 \int_0^x [-\ln(1-\theta)] \mu(ds, d\theta, du).$$

It remains to prove that for all t, all x, $V(t,x) < \infty$ a.s. We will show that its Laplace transform $E(e^{-\lambda V(t,x)})$, which is explicitly computable, tends to 1 as λ tends to 0^+.

$$E(e^{-\lambda V(t,x)}) = \exp\left(-\int_0^t \int_0^1 \int_0^x \{1 - e^{\lambda \ln[1-\theta]}\}(1-\theta)\,\beta(\theta)\,d\theta\right)$$

$$= \exp\left(-tx\int_0^1 [1-(1-\theta)^\lambda](1-\theta)\,\beta(\theta)\,d\theta\right).$$

Recalling that $\int_0^1 \theta(1-\theta)\,\beta(\theta)\,d\theta < \infty$, we deduce that this quantity tends to 1 when λ goes to 0. Indeed, for any $\varepsilon \in]0,1[$, any $\lambda \in]0,1[$,

$$\int_0^1 [1-(1-\theta)^\lambda](1-\theta)\,\beta(\theta)\,d\theta$$

$$\leqslant \int_0^{1-\varepsilon} (1-\varepsilon^\lambda)\,\theta(1-\theta)\,\beta(\theta)\,d\theta + \int_{1-\varepsilon}^1 (1-\theta)\,\beta(\theta)\,d\theta.$$

Choosing $\varepsilon > 0$ small enough and then λ small enough allows to conclude. ∎

We finally exhibit cases where the fragmentation makes X_t reach 0 while $K(0,y) \neq 0$ for all $y \neq 0$.

Proof of Proposition 3.11. This proof is quite the same as that presented for Proposition 3.8. We introduce, for $\varepsilon > 0$ fixed, a C_b^1 increasing concave function ζ^ε, such that $\zeta^\varepsilon(x) = x^{1-\gamma}$ for $x \in [\varepsilon, \infty[$ and such that $(\zeta^\varepsilon)'(x) \leqslant (1-\gamma)/x^\gamma$ on $]0,\infty[$. We compute

$$E[\zeta^\varepsilon(X_t)] = E[\zeta^\varepsilon(X_0)] + \int_0^t E[(\zeta^\varepsilon)'(X_s)\,K(X_s,0)]\,P[X_s=0]\,ds$$

$$+ \int_0^t EE_\alpha\left[\frac{\zeta^\varepsilon(X_s+\tilde{X}_s(\alpha)) - \zeta^\varepsilon(X_s)}{\tilde{X}_s(\alpha)}\,K(X_s,\tilde{X}_s(\alpha))\,\mathbb{1}_{\{\tilde{X}_s(\alpha)>0\}}\right]ds$$

$$- \int_0^t E\left[\frac{1}{X_s}\int_0^{X_s} [\zeta^\varepsilon(X_s) - \zeta^\varepsilon(X_s-y)](X_s-y)\,F(y,X_s-y)\,dy\right]ds.$$

One may check that for any $x \geqslant 0$, $y \geqslant 0$, $\zeta^\varepsilon(x+y) - \zeta^\varepsilon(x) \leqslant y \times (x \vee y)^{-\gamma}$. Using the upper-bound of K yields that for all $x \geqslant 0$, $y \geqslant 0$, $[\zeta^\varepsilon(x+y) - \zeta^\varepsilon(x)]\,K(x,y)/y \leqslant 2$. On the other hand, one clearly has $(\zeta^\varepsilon)'(x)\,K(x,0) \leqslant 1$. Using then the inequality $P(X_s=0) + 2P(X_s>0) \leqslant 2$, the estimates of ψ given in the statement, and the same computations as in the proof of Proposition 3.8, we obtain

$$E[\zeta^{\varepsilon}(X_t)] \leqslant E[\zeta^{\varepsilon}(X_0)] + 2t - \frac{(1-\gamma)\,\rho}{2} \int_0^t E[\mathbb{1}_{\{X_s > 2\varepsilon\}}]\,ds.$$

Since $\zeta^{\varepsilon}(X_t)$ is always nonnegative, we deduce that

$$\int_0^t E[\mathbb{1}_{\{X_s > 2\varepsilon\}}]\,ds \leqslant \frac{2}{(1-\gamma)\,\rho} E[\zeta^{\varepsilon}(X_0)] + \frac{4}{(1-\gamma)\,\rho}\,t.$$

Making ε tend to 0, we finally obtain

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t P[X_s > 0]\,ds \leqslant \frac{4}{(1-\gamma)\,\rho}.$$

This ends the proof. ∎

We finally study the regularization properties of infinite fragmentation. We copy line by line the method of ref. 9 although we can unfortunately not apply directly the results. This was inspired by the works of Bichteler and Jacod.[4] The method below is based on stochastic calculus of variations for jump processes, which was first investigated by Bismut.[5]

We will use the following lemma, which can be found in ref. 9.

Lemma 4.1. Let f be a map from $[a, b]$ into $\mathbb{R}$, which is strongly increasing in the sense that there exists $c > 0$ such that for all $a \leqslant \lambda \leqslant \lambda + v \leqslant b$, $f(\lambda + v) - f(\lambda) \geqslant cv$. Then for all Lebesgue-null set A,

$$\int_a^b \mathbb{1}_A(f(\lambda))\,d\lambda = 0.$$

Proof of Proposition 3.12. We only sketch the proof of point 1. Indeed, point 2 follows by combining Point 1 and Remark 3.10. We fix a terminal time $T > 0$. We first consider the pure fragmentation case ($K = 0$) in Steps 1 to 4, and we extend the result to the case of weak coagulation in Step 5.

Step 1. First of all, note that we may use Remark 3.3:

$$X_t = X_0 - \int_0^t \int_0^1 \int_0^{\infty} \theta X_{s-} \mathbb{1}_{\{u \leqslant X_{s-}\alpha(X_{s-})\}}\,\mu(ds, d\theta, du)$$

where μ has the intensity $ds(1-\theta)\,\beta(\theta)\,d\theta\,du$. Then we may use explicit computation of Doléans–Dade exponentials, see Jacod and Shiryaev,[12] p. 59, to obtain

$$X_t = X_0 \exp(-A_t) \tag{4.11}$$

where

$$A_t = \int_0^t \int_0^1 \int_0^\infty [-\ln(1-\theta)]\, \mathbb{1}_{\{X_{s-}>0\}} \mathbb{1}_{\{u \leqslant X_{s-}\alpha(X_{s-})\}}\, \mu(ds, d\theta, du). \tag{4.12}$$

Step 2. We now consider a C^1 function γ on $[0,1]$, positive on $]0,1[$. Then we use the Girsanov Theorem for point measures, see Jacod and Shiryaev,[12] p. 157. We obtain that if γ and its derivative are sufficiently small (integrability conditions are required), if for all $\lambda \in [0,1]$, the map $\theta \mapsto \theta - \lambda\gamma(\theta)$ is a bijection from $[0,1]$ into itself, then for all $\lambda \in]0,1[$, the law of $\{X_t\}_{t\in[0,T]}$ is absolutely continuous with respect to the law of $\{X_t^\lambda\}_{t\in[0,T]}$, where $\{X_t^\lambda\}_{t\in[0,T]}$ is defined by

$$X_t^\lambda = X_0 - \int_0^t \int_0^1 \int_0^\infty [\theta - \lambda\gamma(\theta)]\, X_{s-}^\lambda\, \mathbb{1}_{\{u \leqslant X_{s-}^\lambda \alpha(X_{s-}^\lambda)\}}\, \mu(ds, d\theta, du).$$

We denote by G_T^λ the corresponding density (which can be explicitly computed). Note that for $\lambda = 0$, we have $X_t^0 = X_t$ for all t. See Definition 4.1 and Proposition 4.2 of ref. 9 for similar arguments.

Step 3. Using the fact that $x \mapsto x\alpha(x)$ decreases, using the exponential expressions of X^λ (as (4.11) and (4.12)), one easily understands that a.s., for each $t \in [0,T]$, the map $\lambda \mapsto X_t^\lambda$ is increasing on $[0,1]$. More precisely, one may prove (following Proposition 5.3 in ref. 9), that for all $0 \leqslant \lambda \leqslant \lambda + v \leqslant 1$,

$$X_t^{\lambda+v} - X_t^\lambda \geqslant v Z_t X_t^\lambda \geqslant 0 \tag{4.13}$$

where

$$Z_t = \int_0^t \int_0^1 \int_0^\infty \gamma(\theta)\, X_{s-}\, \mathbb{1}_{\{u \leqslant X_0 \alpha(X_0)\}}\, \nu(ds, d\theta, du).$$

Since $x\alpha(x) > 0$ for all $x > 0$, since $\gamma(\theta) > 0$ for all $\theta \in]0,1[$, and since $\int_0^1 (1-\theta)\,\beta(\theta)\,d\theta = \infty$, we deduce that

$$\{Z_t > 0\} = \{X_0 > 0\}. \tag{4.14}$$

Step 4. We finally fix $t \in]0,T]$, and consider a Lebesgue-null set $A \subset]0,\infty[$. We have to prove that $P(X_t \in A)=0$. First of all, from Step 2, we have

$$P(X_t \in A) = E(\mathbb{1}_A(X_t^\lambda)\, G_T^\lambda) = \int_0^1 E(\mathbb{1}_A(X_t^\lambda)\, G_T^\lambda)\, d\lambda$$
$$= E\left(\int_0^1 \mathbb{1}_A(X_t^\lambda)\, G_T^\lambda\, d\lambda\right).$$

It thus suffices to prove that a.s., $\int_0^1 \mathbb{1}_A(X_t^\lambda)\, G_T^\lambda\, d\lambda = 0$. Since furthermore $E[G_T^\lambda]=1$ for all λ, we see that a.s., $\int_0^1 G_T^\lambda\, d\lambda < \infty$, and it is enough to check that a.s.,

$$\int_0^1 \mathbb{1}_A(X_t^\lambda)\, d\lambda = 0.$$

We divide Ω in three parts.

(i) First, if $X_t(\omega) = X_t^0(\omega) > 0$, then it is clear that $X_0(\omega) > 0$, and thus (4.14) yields that $Z_t(\omega) > 0$. Hence we know from (4.13) that

$$\inf_{\{0 \leqslant \lambda < \lambda+\nu \leqslant 1\}} \frac{X_t^{\lambda+\nu}(\omega) - X_t^\lambda(\omega)}{\nu} > 0$$

and one may conclude by applying Lemma 4.1 to $f(\lambda) = X_t^\lambda(\omega)$.

(ii) If $X_t^\lambda(\omega) = 0$ for all λ, then the result is obvious since 0 does not belong to A.

(iii) Finally fix ω such that $0 = X_t(\omega) < X_t^1(\omega)$. Denote by $\lambda_0(\omega) = \inf\{\lambda > 0;\, X_t^\lambda(\omega) > 0\}$. Then it is clear that for all $\varepsilon > 0$, $X_t^{\lambda_0(\omega)+\varepsilon}(\omega) > 0$. Using (4.13) shows that

$$\inf_{\{\lambda_0(\omega)+\varepsilon \leqslant \lambda < \lambda+\nu \leqslant 1\}} \frac{X_t^{\lambda+\nu}(\omega) - X_t^\lambda(\omega)}{\nu} > 0$$

and thus one may apply Lemma 4.1 to obtain $\int_{\lambda_0(\omega)+\varepsilon}^1 \mathbb{1}_A(X_t^\lambda(\omega))\, d\lambda = 0$. On the other hand, it is obvious (since $0 \notin A$) that $\int_0^{\lambda_0(\omega)} \mathbb{1}_A(X_t^\lambda(\omega))\, d\lambda = 0$. Making ε tend to 0 allows to conclude.

Step 5. We finally extend this result to the case with coagulation. Denote by $J_t^K = \sum_{s \leqslant t} \mathbb{1}_{\{\Delta X_s > 0\}}$ the number of jumps of X due to the coagulation before time t:

$$J_t^K = \int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\left\{u \leqslant \frac{K(X_{s-},\, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds, d\alpha, du).$$

Since $K(x, y) \leqslant Cxy$, one easily checks that $E[J_t^K] < \infty$ for all $t \geqslant 0$. Denote by $0 < T_1^K < T_2^K < \cdots$ the corresponding instants. Then $\bigcup_{i \geqslant 1} \{T_i^K\} \subset \bigcup_{i \geqslant 1} \{S_i\}$, where we define the instants S_i recursively by $S_0 = 0$, and

$$S_{n+1} = \inf \left\{ s > S_n ; \int_{S_n}^{s} \int_0^1 \int_0^\infty \mathbb{1}_{\{u \leqslant C X_{S_n}\}} N(ds, d\alpha, du) \geqslant 1 \right\}.$$

This comes from the fact that $K(x, y)/y \leqslant Cx$, and that the process X_t does not increase between two instants of coagulation. We thus may split our process X_t in the following way:

$$X_t = \sum_{i \geqslant 0} X^i_{t - S_i} \mathbb{1}_{[S_i, S_{i+1}[}(t)$$

where for each i, conditionally to $\mathscr{G}_{S_i}$, X_t^i is a pure fragmentation process, starting at X_{S_i} and independent of S_{i+1}. We now may conclude: consider a Lebesgue-null subset A of $]0, \infty[$. Then, for all $t > 0$,

$$\begin{aligned} P[X_t \in A] &= \sum_{i \geqslant 0} P(X^i_{t - S_i} \in A, t \in]S_i, S_{i+1}[) \\ &\leqslant \sum_{i \geqslant 0} P(X^i_{t - S_i} \in A, t > S_i) = 0 \end{aligned}$$

by the four first steps. This concludes the proof. ∎

REFERENCES

1. D. J. Aldous, Deterministic and stochastic models for coalescence (aggregation, coagulation): A review of the mean-field theory for probabilists, *Bernoulli* **5**:3–48 (1999).
2. J. Bertoin, Homogeneous fragmentation processes, *Probab. Theory Related Fields* **121**: 301–318 (2001).
3. J. Bertoin, *Self-Similar Fragmentations*, prépublication 610 du laboratoire de probabilités de Paris 6.
4. K. Bichteler and J. Jacod, Calcul de Malliavin pour les diffusions avec sauts: existence d'une densité dans le cas unidimensionnel, *Séminaire de Probabilités XVII, L.N.M. 986* (Springer, 1983), pp. 132–157.
5. J. M. Bismut, Calcul des variations stochastique et processus de sauts, *Z. Wahrsch. Verw. Gebiete* **63**:147–235 (1983).
6. M. Deaconu and E. Tanré, Smoluchowski's coagulation equation: Probabilistic interpretation of solutions for constant, additive, and multiplicative kernels, *Annali della Scuola Normale Superiore di Pisa, Serie IV*, Vol. XXIX, 3, 2000.
7. M. Deaconu, N. Fournier, and E. Tanré, A pure jump Markov process associated with Smoluchowski's coagulation equation, to appear in *Ann. Probab*.
8. N. Fournier and S. Méléard, A weak criterion of absolute continuity for jump processes, to appear in *Bernoulli.*

9. N. Fournier, Jumping SDEs: Absolute continuity using monotonicity, *Stochastic Process. Appl.* **98**:317–330 (2002).
10. C. Graham and S. Méléard, Existence and regularity of a solution to a Kac equation without cutoff by using the stochastic calculus of variations, *Commun. Math. Phys.* **205**: 551–569 (1999).
11. B. Haas, *Loss of Mass in Deterministic and Random Fragmentations*, prépublication 730 du laboratoire de probabilités de Paris 6.
12. J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes* (Springer, 1987).
13. B. Jourdain, *Nonlinear Processes Associated with the Discrete Smoluchowski Coagulation Fragmentation Equation*, Prépublication du CERMICS, 207, 2001.
14. P. Laurençot, On a class of coagulation fragmentation equations, *J. Differential Equations* **167**:245–274 (2000).
15. J. R. Norris, Smoluchowski's coagulation equation: uniqueness, non-unique-ness and hydrodynamic limit for the stochastic coalescent, *Ann. Appl. Probab.* **9**:78–109 (1999).
16. H. Tanaka, Probabilistic treatment of the Boltzmann equation of Maxwellian molecules, *Z. Wahr. Verw. Gebiete* **46**:67–105 (1978).